International Journal of Theoretical Physics, Vol. 46, No. 6, June 2007 (© 2007)
DOI: 10.1007/s10773-006-9305-y

Transformation Properties of Dynamical Systems at the Quantum Level

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Received September 22, 2006; accepted November 9, 2006
Published Online: February 21, 2007

Based on the generating functional of Green function for a dynamical system, the general equations of transformation properties at the quantum level are derived. In some cases they can be reduced to the quantum Noether theorem. In some other cases they can be reduced to momentum theorem or angular momentum theorem etc. at the quantum level. An example is presented and it shows that the classical conservation laws don't always preserve in quantum theories.

KEY WORDS: symmetry and conservation laws; constrained Hamiltonian systems; path integral quantization.

PACS: 11.10.E

1. INTRODUCTION

D Symmetry is a fundamental concept in modern physics. The connection between the symmetry of the action integral under finite continuous group and conservation law is given by the first (classical) Noether theorem. The second Noether theorem refers to the invariance of the action under an infinite continuous group. In this case there exist differential identities, which involve variational derivatives of the action and are called Noether identities (Noether, 1918). These theorems have an important role in theoretical physics. A generalization of the Noether theorem was given by Rosen and others (1974). A generalization of the Noether theorem for nonholonomic systems and a generalization of Noether identities for variant systems were studied by one of the authors and others (Li, 1981, 1987, 1992; Li and Li, 1990). The general achievements have been made in the study of symmetries, however, there are also some questions open. First, whether the connection between symmetries and conservation laws is hold true in quantum theory? Although it has been discussed in literatures (Li, 1996; Li and

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Gao, 1997), we shall further study the problem here; Second, what results can be derived when a system is not invariant under infinitesimal transformations at the quantum level? We shall discuss the transformation properties of a dynamical system at the quantum level.

In order to study the quantum properties of a system, the first step is to formulate its quantization. One method to quantize a system is the canonical quantization, another one known as the Feymann path integral. The advantage of the path integral scheme to discuss the transformation properties of the system at the quantum level is that we dealt with c-number, not q-number. In this paper, based on the path integral formalism we consider the change of the action integral under the infinitesimal transformation and the corresponding change of the integral measure of the path integral, and we can find the general equations of transformation properties at the quantum level. In some cases they can be reduced to the quantum Noether theorem. In some other cases they can be reduced to momentum theorem or angular momentum theorem etc. in quantum mechanics. It shows that the connection between symmetries and conservation laws in classical theories may be not hold true at the quantum level when the integral measure is not invariance under the transformation.

2. A SYSTEM WITH A REGULAR LAGRANGIAN

For the sake of simplicity we deal with a system with a finite number of degrees of freedom exhibiting the essential properties of theories, and the extension to field theories is formally straightforward. We first consider a system described by a regular Lagrangian $L(q, \dot{q})(q = [q^1, q^2, \ldots, q^n])$, and the generalized momenta conjugate to generalized coordinates q are denoted by $p = \partial L/\partial\dot{q}(p = [p_1, p_2, \ldots, p_n])$. According to Feymann path integral quantization, the transformation function which describes transition matrix element of the particle from initial state to final state can be written as

$$Z[0] = \int \mathrm{D}q\mathrm{D}p \exp\left\{i\int dt[p\dot{q} - H_c]\right\} \tag{1a}$$

where H_c is a canonical Hamiltonian. Introducing the exterior sources $J = (J_1, J_2, \ldots, J_n)$ with respect to the generalized coordinates $q = [q^1, q^2, \ldots, q^n]$, respectively, we can write the generating functional of Green function in phase space for this system as (Gtman and Tyutin, 1990)

$$Z[J] = \int \mathrm{D}q\mathrm{D}p \exp\left\{i\int dt[p\dot{q} - H_c + Jq]\right\} \tag{1b}$$

From (1b) we can derive all the quantities of interest at the quantum level. Let us consider the case in which the canonical Hamiltonian is quadratic in momenta, for example, $H_c = \frac{1}{2}p^2 + V(q)$. The integration over momenta in (1b) can be performed, and we get (Gtman and Tyutin, 1990)

$$Z[J] = \int \mathrm{D}q \exp\left\{i \int dt[L_{eff} + Jq]\right\} \tag{2}$$

where L_{eff} is an effective Lagrangian in configuration space, and $L_{eff} = L$ in this case. The action of the system is given by $I = I_{eff} = \int L_{eff}dt$.

Consider the following global transformation

$$\begin{cases} t' = t + \Delta t = t + \varepsilon_\sigma \tau^\sigma(t;q) \\ q'(t') = q(t) + \Delta q(t) = q(t) + \varepsilon_\sigma \xi^\sigma(t;q) \end{cases} \tag{3a}$$

where $\varepsilon_\sigma(\sigma = 1, 2, \ldots, r)$ are infinitesimal arbitrary parameters, and it be supposed that the action of the system is not invariant under the transformation (3a), and the corresponding variation of the action is given by

$$\delta I = \int u^\sigma(q, \dot{q})\varepsilon_\sigma dt \tag{4}$$

We localize the transformation (3a), and we get the corresponding local transformation

$$\begin{cases} \bar{t} = t + \Delta t = t + \varepsilon_\sigma(t)\tau^\sigma(t;q) \\ \bar{q}(\bar{t}) = q(t) + \Delta q(t) = q(t) + \varepsilon_\sigma(t)\xi^\sigma(t;q) \end{cases} \tag{3b}$$

where $\varepsilon_\sigma(t)(\sigma = 1, 2, \ldots, r)$ are infinitesimal arbitrary functions, and their values and derivatives will vanish at the end-points $t = t_1$ and $t = t_2$. The change of the action under the local transformation (3b) is given by Li and Jiang (2002)

$$\begin{aligned} \Delta I &= \int dt\left\{\frac{\delta I}{\delta q}(\Delta q - \dot{q}\Delta t) + D\left[\frac{\partial L}{\partial \dot{q}}(\Delta q - \dot{q}\Delta t) + L\Delta t\right]\right\} \\ &= \int dt\varepsilon_\sigma(t)\left\{\frac{\delta I}{\delta q}(\xi^\sigma - \dot{q}\tau^\sigma) + D\left[\frac{\partial L}{\partial \dot{q}}(\xi^\sigma - \dot{q}\tau^\sigma) + L\tau^\sigma\right]\right\} \\ &\quad + \int dt\left\{\left[L\tau^\sigma + \frac{\partial L}{\partial \dot{q}}(\xi^\sigma - \dot{q}\tau^\sigma)\right]D\varepsilon_\sigma(t)\right\} \end{aligned} \tag{5}$$

where $D = d/dt$, and

$$\frac{\delta I}{\delta q} = \frac{\partial L}{\partial q} - D\left(\frac{\partial L}{\partial \dot{q}}\right) \tag{6}$$

Since the variation of the action under the global transformation (3a) is given by (4), hence the first integral in (5) is equal to (4). According to the boundary conditions of $\varepsilon_\sigma(t)$, from (5) we get

$$\Delta I = -\int dt\left\{\varepsilon_\sigma(t)D\left[\frac{\partial L}{\partial \dot{q}}(\xi^\sigma - \dot{q}\tau^\sigma) + L\tau^\sigma\right] - u^\sigma\varepsilon_\sigma(t)\right\} \tag{7}$$

Let it be supposed that the Jacobian of the transformation (3b) is $\bar{J} = 1 + J_1(\varepsilon)$. The generating functional (2) is invariant under the transformation (3b), and we obtain

$$Z[J] = \int \mathrm{D}q\bar{J}\left\{1 - i\int dt\varepsilon_\sigma(t)\left\{D\left[L\tau^\sigma + \frac{\partial L}{\partial\dot{q}}(\xi^\sigma - \dot{q}\tau^\sigma)\right] - u^\sigma\right.\right.$$
$$\left.\left. - J(\xi^\sigma - \dot{q}\tau^\sigma)\right\}\right\} \bullet \exp\left\{i\int dt(L + Jq)\right\}$$
$$= \int \mathrm{D}q\left\{1 + J_1 - i\int dt\varepsilon_\sigma(t)\left\{D\left[L\tau^\sigma + \frac{\partial L}{\partial\dot{q}}(\xi^\sigma - \dot{q}\tau^\sigma)\right] - u^\sigma\right.\right.$$
$$\left.\left. - J(\xi^\sigma - \dot{q}\tau^\sigma)\right\}\right\} \bullet \exp\left\{i\int dt(L + Jq)\right\} \tag{8}$$

Differentiating functionally (8) with respect to $\varepsilon_\sigma(t)$, we have

$$\int \mathrm{D}q\left\{D\left[L\tau^\sigma + \frac{\partial L}{\partial\dot{q}}(\xi^\sigma - \dot{q}\tau^\sigma)\right] - u^\sigma - J(\xi^\sigma - \dot{q}\tau^\sigma) - J^\sigma\right\}$$
$$\exp\left\{i\int dt(L + Jq)\right\} = 0 \tag{9}$$

where $J^\sigma = -i\delta\bar{J}/\delta\varepsilon_\sigma(t)|_{\varepsilon=0}$. Let $J = 0$ in (9), and we obtain

$$\left\langle 0\left|T^*\left\{D\left[L\tau^\sigma + \frac{\partial L}{\partial\dot{q}}(\xi^\sigma - \dot{q}\tau^\sigma)\right] - u^\sigma - J^\sigma\right\}\right|0\right\rangle = 0 \tag{10}$$

where $|0\rangle$ is the ground state of the quantum system, and T^* stands for the covariantized T product (Young, 1987) in which derivatives of operators inside a T-product are defined in terms of the dipole formula, i.e., $\langle 0|\, T^*[D_t q(t) D_{t'} q(t')\ldots]\,|0\rangle = D_t D_{t'}\,\langle 0|\, T[q(t)q(t')\ldots]\,|0\rangle$. For a system with a non-invariance action and integral measure under the transformation (3b), we can obtain the general equations of transformation properties at the quantum level

$$\left\langle 0\left|T^*\left\{D\left[L\tau^\sigma + \frac{\partial L}{\partial\dot{q}}(\xi^\sigma - \dot{q}\tau^\sigma)\right]\right\}\right|0\right\rangle = \langle 0|\, T^*[u^\sigma + J^\sigma]\,|0\rangle \tag{11}$$

When the action of the system is invariant under the transformation (3b) and the corresponding Jacobian of this transformation is equal to unity, i.e., $u^\sigma = 0$, $J^\sigma = 0$, from (11), we get

$$\left\langle 0\left|T^*\left\{D\left[L\tau^\sigma + \frac{\partial L}{\partial\dot{q}}(\xi^\sigma - \dot{q}\tau^\sigma)\right]\right\}\right|0\right\rangle = 0 \tag{12a}$$

i.e.,

$$\left\langle 0 \left| T\left[L\tau^{\sigma} + \frac{\partial L}{\partial \dot{q}}(\xi^{\sigma} - \dot{q}\tau^{\sigma}) \right] \right| 0 \right\rangle = \text{const} \tag{12b}$$

Thus, we have the following Noether theorem at the quantum level: If the action of a system and the integral measure of the generating functional are invariant under the transformation (3b), then, there are some conserved quantities (12b) at the quantum level.

When the action of the system is invariant under the transformation (3b) and the corresponding Jacobian is not equal to unity, i.e., $u^{\sigma} = 0$, $J^{\sigma} \neq 0$, from (11), we get

$$\left\langle 0 \left| T^{*}\left\{ D\left[L\tau^{\sigma} + \frac{\partial L}{\partial \dot{q}}(\xi^{\sigma} - \dot{q}\tau^{\sigma}) \right] \right\} \right| 0 \right\rangle = \langle 0|T^{*}[J^{\sigma}]|0\rangle \neq 0 \tag{13}$$

Thus, in this case the result of the classical Noether theorem is not hold true at the quantum level when the integral measure of the path integral is not invariant under the transformation. The anomalies could be viewed as a result of the non-invariance of the functional measure under some symmetry transformation (Fujikaw, 1980).

When the action of the system is not invariant under the transformation (3b) and the corresponding Jacobian is not equal to unity, i.e., $u^{\sigma} \neq 0$, $J^{\sigma} \neq 0$, but $u^{\sigma} + J^{\sigma} = 0$, from (11) we also get

$$\left\langle 0 \left| T\left[L\tau^{\sigma} + \frac{\partial L}{\partial \dot{q}}(\xi^{\sigma} - \dot{q}\tau^{\sigma}) \right] \right| 0 \right\rangle = \text{const} \tag{14}$$

There may also have conserved quantities in quantum theories when the action of a system is not invariant under some transformations. In general, there are no corresponding conserved quantities in classical theories.

3. A SYSTEM WITH A SINGULAR LAGRANGIAN

A dynamical system described by a singular Lagrangian is subject to some inherent phase space constraints and is called a constrained Hamilton system. The motion of the system is restricted to a hypersurface of the phase space determined by the phase space constraints. The quantization of the systems with constraints based on Dirac's theory and using the method of path (functional) integral was performed by Faddeev (1970). Faddeev restricted his discussion to the case when only first-class constraints are present. A generalization of Faddeev's method to a system which contains second-class constraints was done by Senjanovic (1976). In general, the gauge conditions are not relativistic covariant in the above methods. Based on BRST (Becchi-Rouet-Stora-Tyutin) symmetries, the relativistic covariant quantization method was developed by Batalin, Fradkin and Vilkovsky, and is called BFV quantization (Henneaux, 1985).

Let $\Lambda_k(q, p) \approx 0(k = 1, 2, \ldots, K)$ be first-class constraints, and $\theta_i(q, p) \approx 0(i = 1, 2, \ldots, I)$ be second-class constraints. The gauge conditions connecting with the first-class constraints are $\Omega_k(q, p) \approx 0(k = 1, 2, \ldots, K)$. According to Faddeev-Senjanovic path integral formalism (Faddeev, 1970; Senjanovic, 1976), the generating functional of Green function in phase space for this system is given by

$$Z[J] = \int \mathrm{D}q\mathrm{D}p \prod_{j,k,l} \delta(\theta_j)\delta(\Lambda_k)\delta(\Omega_l) \det |\{\Lambda_k, \Omega_l\}| \cdot [\det |\{\theta_i, \theta_j\}|]^{1/2}$$
$$\exp\left\{i \int dt[p\dot{q} - H_c + Jq]\right\} \tag{15a}$$

where H_c is the canonical Hamiltonian, and $J = (J_1, J_2, \ldots, J_n)$ are the exterior sources with respect to $q = [q^1, q^2, \ldots, q^n]$, and $\{.,.\}$ stands for Poisson bracket. Using the integral properties of the Grassmann variables and δ-functions, the expression (15a) can be written as (Li and Jiang, 2002)

$$Z[J] = \int \mathrm{D}q\mathrm{D}p \exp\left\{i \int dt\left[L_{eff}^p + Jq\right]\right\} \tag{15b}$$

where L_{eff}^p is the effective Lagrangian in phase space (Li and Jiang, 2002). When the momentum is quadratic in (15b), then the integration over$p$ can be performed, and we obtain

$$Z[J] = \int \mathrm{D}q \exp\left\{i \int dt[L_{eff} + Jq]\right\} \tag{16}$$

where L_{eff} is the effective Lagrangian in configuration space. For a gauge-invariant system, we can use also the Faddeev-Popov method to quantize the system in configuration space (Young, 1987). In certain case (for example, Yang-Mills theory), according to the path-integral quantization of constrained Hamiltonian systems, we can carry out explicit integration over the momenta in the phase-space path integral and get the same results obtained by using Faddeev-Popov method (Gtman and Tyutin, 1990). For a system with a singular Lagrangian, in this case we can still proceed in the same way as with a regular Lagrangian to study the transformation properties at the quantum level.

In general, the momenta in (1b) or (15b) is not quadratic, thus we can not get the generating functional of Green function in configuration space. In these cases, we can discuss the transformation properties of dynamical systems in phase space at the quantum level.

4. AN EXAMPLE

Consider a system with a regular Lagrangian

$$L = \frac{1}{2}\dot{q}^2 - V(q) \tag{17}$$

After the integration over the momenta in the path integral (1b), the generating functional of Green function in configuration space for a system with Lagrangian (17) is given by

$$Z[J] = \int \mathrm{D}q \exp\left\{i \int dt[L_{eff} + Jq]\right\} \tag{18}$$

here $L_{eff} = L$. Let us assume that q are Cartesian coordinates x in three dimensional space. First, consider the following infinitesimal space translation transformation

$$\begin{cases} q'(t') = q(t) + \varepsilon \\ t' = t \end{cases} \tag{19}$$

where ε is an infinitesimal arbitrary parameter. The variation of the action of the system under the transformation (19) is given by $\delta I = \int_{t_1}^{t_2} (-\frac{\partial V}{\partial q})\varepsilon\, dt = \int_{t_1}^{t_2} F\varepsilon\, dt$ and the Jacobian of the transformation (19) is equal to unity. It is easy to see that $\tau^\sigma = 0,\ \xi = 1$in (19) according to (3b), thus, from (11) we have

$$D\,\langle 0|\, p\, |0\rangle = \langle 0|\, F\, |0\rangle \tag{20}$$

i.e.,

$$\langle 0|\, [p_2 - p_1]\, |0\rangle = \int_{t_1}^{t_2} \langle 0|\, F\, |0\rangle\, dt \tag{21}$$

This is the momentum theorem in quantum mechanics.

Second, consider an infinitesimal space rotation transformation

$$\begin{cases} q_i'(t) = q_i + \varepsilon_{ij} q_j \\ t' = t \end{cases} \qquad (|\varepsilon_{ij}| \ll 1) \tag{22}$$

where ε_{ij} are infinitesimal arbitrary parameters. The variation of the action under the transformation (22) is given by

$$\delta I = \int_{t_1}^{t_2} \left(-\frac{\partial V}{\partial q_i} \varepsilon_{ij} q_j \right) dt = \int_{t_1}^{t_2} \varepsilon_{ij} q_j F_i dt \tag{23}$$

and the Jacobian of the transformation (22) is equal to unity. It is easy to see that $\tau^\sigma = 0, \xi_i = q_j$ in (22), thus, from (11) we have

$$\langle 0|\, T[p_i q_j - p_j q_i]_{t=t_1}^{t=t_2}\, |0\rangle = \int_{t_1}^{t_2} \langle 0|\, T[q_j F_i - q_i F_j]\, |0\rangle dt \tag{24}$$

This is the angular momentum theorem in quantum mechanics.

Third, consider an infinitesimal time translation transformation

$$\begin{cases} q'(t') = q(t) \\ t' = t + \varepsilon \end{cases} \tag{25}$$

where ε is an infinitesimal arbitrary parameter. The action of the system is invariant under the transformation (25), and the Jacobian of the transformation (25) is equal to unity. It is easy to see that $\tau = 1, \xi = 0$ in this case, thus, from (12b) we have the conserved energy at the quantum level

$$\left\langle 0 \middle| T\left[L - \frac{\partial L}{\partial \dot{q}}\dot{q}\right] \middle| 0 \right\rangle = -\left\langle 0 \middle| T\left[\frac{1}{2}\dot{q}^2 + V(q)\right] \middle| 0 \right\rangle = \text{const} \tag{26}$$

Finally, suppose the following condition (Bobillo-Ares, 1988)

$$V(aq) = a^{-2}V(q) \tag{27}$$

is satisfied in (17), where a is an arbitrary parameter. Consider the following transformation (Bobillo-Ares, 1988)

$$\begin{cases} q' = q + \varepsilon q \\ t' = t + 2\varepsilon t \end{cases} \tag{28}$$

where ε is an infinitesimal arbitrary parameter. The action of the system is invariant under the transformation (28), but the Jacobian of the transformation (28) is not equal to unity. From (13), it is easy to see that there is no conserved quantity corresponding to the transformation (28) at the quantum level. But based on the classical Noether theorem there exists the corresponding conserved quantity (Bobillo-Ares, 1988).

5. CONCLUSIONS

In conclusion, we consider the changes of the action integral and the integral measure of the generating functional under the infinitesimal transformation in the path integral formalism, and we obtain the general equations of transformation properties at the quantum level. In some cases they can be reduced to the quantum Noether theorem. In some other cases they can be reduced to momentum theorem or angular momentum theorem etc. at the quantum level. It shows that the classical conserved laws are not always hold true at the quantum level when the integral measure in path integral is not equal to unity under the transformation (3b). When we take the change of integral measure into account, for a system which is not invariant under the transformation (3a), there may also have some conserved quantities at the quantum level if the condition $J^\sigma + u^\sigma = 0$ is satisfied. But, in general, there are no corresponding classical conserved charges in this case.

REFERENCES

Bobillo-Ares, N. (1988). *American Journal of Physics* **56**, 174.
Faddeev, L. D. (1970). *Theoretical Mathematical Physics* **1**, 1.
Fujikaw, K. (1980). *Physics Review Letter* **44**, 1733.
Gtman, D. M. and Tyutin, I. V. (1990). *Quantization of Fields with Constraints*, Springer-Verlag, Berlin.
Henneaux, M. (1985). *Physics Report* **126**, 1.
Li, Z. P. (1981). *Acta Physics Sinica* **30**, 1599.
Li, Z. P. (1992). *Acta Mathematical Science* **12**, 241.
Li, Z. P (1987). *International Theoretical Physics* **26**, 853.
Li, Z. P. (1996). *Science in China* **A26**, 649 (in Chinese).
Li, Z. P. (1996). *Acta Physics Sinica* **45**, 1601(in Chinese).
Li Z. P. and Gao, H. X. (1997). *International Journal of Theoretical Physics* **36**, 1071.
Li, Z. P. and Jiang, J. H. (2002). *Symmetries in Constrained Canonical Systems*, Science Press, Beijing.
Li, Z. P. and Li, X (1990). *International Journal of Theoretical Physics* **29**, 765.
Noether, E. (1918). Nachrichten Akad. Wissenschaft, Gottingen, Mathematical Physics Kl 235.
Rosen, J. (1974). *Annals Physics* (NY) **82**, 54.
Senjanovic, P. (1976). *Annal Physics* **100,** 227.
Young, B. L. (1987). *Introduction to Quantum Field Theories*, Science Press, Beijing.